

A variant β -Wythoff Nim on Beatty's theorem

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Abstract

We give short rules for two-pile take-away games satisfying that a pair of complementary homogeneous Beatty sequences together with $(0, 0)$ constitute a complete set of P -positions.

A NEW CONSTRUCTION STRICTLY IN BETWEEN NIM AND k -WYTHOFF NIM. Let us recall the rules of k -Wythoff Nim [F], k a positive integer. The available positions are (x, y) , x and y non-negative integers. The legal moves are

(1A) Nim type: $(x, y) \rightarrow (x - t, y)$, if $x - t \geq 0$ and $(x, y) \rightarrow (x, y - t)$, if $y - t \geq 0$.

(1B) Extended diagonal type: $(x, y) \rightarrow (x - s, y - t)$ if $|s - t| < k$ and $x - s \geq 0, y - t \geq 0$.

Hence, this game is a so-called impartial take-away game [WW]. We play normal play, that is the last player to move wins. By the rules (1A) and (1B) we note that k -Wythoff Nim is a so-called ‘invariant’ [DR, LHF] take-away game, that is, each available move is legal from any position as long as the resulting position has non-negative coordinates.

In this note we study another type of take away games, where certain positions have some local restrictions on the set of otherwise ‘invariant’ moves. Such games are sometimes called ‘variant’, eg. [DR, LHF].

Example 1 As usual, let $\pi = 3, 14 \dots$ denote the ratio of the circumference of a circle to its diameter. Let the game rules of π -Wythoff Nim (our notation) be as in k -Wythoff Nim with $k = \lfloor \pi \rfloor = 3$, except that if a player plays from a position where one of the coordinates equals $\lfloor \pi n \rfloor$, for some $n \in \mathbb{N}$, then only Nim type moves (1A) are allowed. This latter rule clearly makes π -Wythoff Nim a ‘variant’ game. Denote with $\alpha = \pi/(\pi - 1)$. Then clearly both α and π are irrational, so that, by Beatty's/Rayleigh's [B,R] theorem,

the sequences $(\lfloor n\alpha \rfloor)$ and $(\lfloor n\pi \rfloor)$ are complementary on the positive integers. The main theorem of this note says that the P -positions of π -Wythoff Nim are identical to the set $\{(\lfloor n\alpha \rfloor, \lfloor n\pi \rfloor), (\lfloor n\pi \rfloor, \lfloor n\alpha \rfloor) \mid n \in \mathbb{Z}_{\geq 0}\}$.

In general, fix an irrational $2 < \beta$. Then play the following variant game on the pairs of non-negative integers: The moves are as in k -Wythoff Nim with $k = \lfloor \beta \rfloor$ (1A) and (1B), except if one of the coordinates is of the form $\lfloor \beta n \rfloor$, $n \in \mathbb{Z}_{>0}$, then only Nim-type moves (1A) are allowed. Denote this game by β -Wythoff Nim.

Main Theorem The P -positions of β -Wythoff Nim are

$$\{(\lfloor n\alpha \rfloor, \lfloor n\beta \rfloor), (\lfloor n\beta \rfloor, \lfloor n\alpha \rfloor) \mid n \in \mathbb{Z}_{\geq 0}\},$$

where n ranges over the non-negative integers and where $\alpha = \beta/(\beta - 1)$.

Proof. $P \rightarrow N$: We have to prove that from each position of the form

$$(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor), \tag{1}$$

there is no move to a position of the same form, or to its symmetric counterpart of the form $(\lfloor \beta n \rfloor, \lfloor \alpha n \rfloor)$. So, suppose that we play from a position of the form in (1). Then, by the rules of game, only (1A) Nim type moves are allowed so that, by complementarity [B,R], there is no move to a position of the same form.

$N \rightarrow P$: If the candidate N -position (x, y) , $x \leq y$, has a coordinate of the form $\lfloor \beta n \rfloor$ then we have to show that a Nim type (1A) move suffices. If $x = \lfloor \beta n \rfloor$ then move $(x, y) \rightarrow (\lfloor \beta n \rfloor, \lfloor \alpha n \rfloor)$. If $y = \lfloor \beta n \rfloor$ and $x > \lfloor \alpha n \rfloor$ move $(x, y) \rightarrow (\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor)$. If $y = \lfloor \beta n \rfloor$ and $x < \lfloor \alpha n \rfloor$, by complementarity [B,R], there is an $m < n$ such that there is a Nim type (1A) move of precisely one of the forms $(x, y) \rightarrow (\lfloor \alpha m \rfloor, \lfloor \beta m \rfloor)$ or $(x, y) \rightarrow (\lfloor \beta m \rfloor, \lfloor \alpha m \rfloor)$. (We use that $\lfloor \alpha m \rfloor \leq \lfloor \beta m \rfloor < \lfloor \beta n \rfloor$.)

Otherwise, a Nim type move obviously suffices if $\lfloor \alpha m \rfloor = y > \lfloor \beta n \rfloor$ for some $n < m$ and $x = \lfloor \alpha n \rfloor$, so suppose that

$$\lfloor \alpha m \rfloor = y < \lfloor \beta n \rfloor \text{ and } x = \lfloor \alpha n \rfloor. \tag{2}$$

(Still with $m > n$.) For this case, by complementarity, a Nim type move to a candidate P -position does not exist, so we have to find a (1B) “extended diagonal” type move.

Note that an “ordered vector subtraction” of consecutive candidate P -positions gives an expression of the form

$$(\lfloor \alpha n \rfloor, \lfloor \beta n \rfloor) - (\lfloor \alpha(n-1) \rfloor, \lfloor \beta(n-1) \rfloor),$$

$n \in \mathbb{Z}_{>0}$, which equals precisely one of the four ordered pairs of differences:

$$(1, \lfloor \beta \rfloor),$$

$$(1, \lfloor \beta + 1 \rfloor),$$

$$(2, \lfloor \beta \rfloor),$$

or

$$(2, \lfloor \beta + 1 \rfloor).$$

The difference of the coordinates in such a pair of differences is bounded by $\pm \lfloor \beta \rfloor$. By our assumption (2) this gives that there is a type (1B) move to a position of the form of a candidate P -position, $(\lfloor \alpha p \rfloor, \lfloor \beta p \rfloor)$, $p < n$. In fact, a “worst case scenario” would be from an N -position of the form $(x, y) = (\lfloor \alpha n \rfloor + t, \lfloor \beta n \rfloor - 1 + t)$, $t \in \mathbb{Z}_{\geq 0}$, together with the above second case difference pair, $(1, \lfloor \beta + 1 \rfloor)$. But, indeed, here a move of type (1B) suffices to the P -position $(\lfloor \alpha(n-1) \rfloor, \lfloor \beta(n-1) \rfloor)$. \square

QUESTIONS AND REMARKS. Suppose that we fix a β and then increase the density of the pairs of sequences from 1 to say an arbitrary number $\gamma > 1$ (or decreases to a density < 1) where α is defined via $1/\alpha + 1/\beta = \gamma$. Given candidate P -positions as above (Main Theorem), is there still a “succinct” and non-trivial way of formulating the game rules without revealing both irrationals or/and the joint density of the sequences? As a remark, observe that neither α nor the density 1 is given away in the presentation of the rules of β -Wythoff Nim. In [LHF] invariant game rules are given for candidate P -positions constructed from complementary Beatty sequences, but not in a single case have we found a “succinct” description. In this note we have chosen to remove the nice condition of invariance from the game rules and, maybe even more notably, one of the coordinates of the candidate P -positions

is revealed within the game rules. This could be argued to be a severe drawback in a definition of the rules of a game. But, on the other hand, we were able to give a very succinct formulation, without a complete trivialization of game rules, for all complementary Beatty sequences and these are uncountably many.

Bibliography

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